Discrete quantum groups

A. Van Daele

Department of Mathematics
KU Leuven (Belgium)

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Introduction

Discrete quantum groups were first introduced in a paper by Podleś and Woronowicz "Quantum deformation of Lorentz group" (1990). Later they were treated by Effros and Ruan (1994) and myself (1996) independently.

The duality between compact quantum groups and discrete quantum groups is a special case of the more general duality of multiplier Hopf algebras with integrals as obtained by myself in "An algebraic framework for group duality " (1998). This theory was the basis of the development by Kustermans and Vaes of the theory of locally compact quantum groups.

Looking at a discrete quantum group as the dual of a compact quantum group is somewhat unnatural. One should approach them as quantizations of discrete groups in their own right. Doing this has some advantages.

Unfortunately, also in recent papers on discrete quantum groups, this point of view has not really found its way. The reason is probably that compact quantum groups were developed prior to discrete quantum groups and that people were familiar with the theory of compact quantum groups before this.

Discrete quantum groups

We will work with the following definition.

Definition

Let I be an index set. For each $\alpha \in I$ we have a full matrix algebra A_α with the usual *-structure. We let A be the direct sum of these components. We consider A_α as sitting in A.

The multiplier algebra $M(A)$ is the direct product of the algebras A_α. Elements a in $M(A)$ are written as $(a_\alpha)_{\alpha \in I}$ where $a_\alpha \in A_\alpha$ for all α. Such an element a belongs to A if and only if only all but finitely many components are non-zero.

The algebra $A \otimes A$ is the algebraic tensor product and $M(A \otimes A)$ its multiplier algebra. Elements a in $M(A \otimes A)$ are characterized by their components $a_{\alpha\beta}$. Such an element belongs to $A \otimes A$ if and only if only finitely many components are non-zero.

The main definition

Definition

A coproduct on A is a *-homomorphism $\Delta : A \to M(A \otimes A)$. It is assumed that the sets

$$\Delta(A)(1 \otimes A) \quad \text{and} \quad (A \otimes 1)\Delta(A)$$

belong to $A \otimes A$ and that Δ is coassociative.

Definition

The pair (A, Δ) is a discrete quantum group if there is a counit and an antipode.

The counit ε is characterized by the property that, for all a, b, c,

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab \quad \text{and} \quad (\iota \otimes \varepsilon)((c \otimes 1)\Delta(a)) = ca.$$

For the antipode S we require for all a, b, c,

$$m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b \quad \text{and} \quad m(\iota \otimes S)((c \otimes 1)\Delta(a)) = c\varepsilon(a).$$

The cointegral

Proposition

There is a self-adjoint idempotent h in A satisfying $ah = \varepsilon(a)h$ for all $a \in A$. It is unique with this property.

The kernel of ε is a two-sided *-ideal of codimension 1. The counit is supported on a one-dimensional component A_e and h is the identity in this component.

Proposition

We have for all $a \in A$

$$\Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes S(a)) \quad \textit{and} \quad (1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h).$$

Properties of the antipode

Because the antipode is an anti-isomorphism and $S(a)^* = S^{-1}(a^*)$, we have

Proposition

For each index α there is an index $\overline{\alpha}$ satisfying

$$S(A_\alpha) = A_{\overline{\alpha}} \quad \textit{and} \quad S(A_{\overline{\alpha}}) = A_\alpha .$$

So S^2 is an isomorphism of each component.

From this result we get

Proposition

Let ω be a trace on A and $q = (\omega \otimes \iota)\Delta(h)$. Then $q \in M(A)$ and $aq = qS^2(a)$ for all a .

If ω is faithful, then q is invertible in $M(A)$ and if ω is positive then q is positive.

The separability idempotent $\Delta(h)$

Fix an index α and consider $\overline{\alpha}$. Let q be a positive invertible element in A_α satisfying $aq = qS^2(a)$ for all $a \in A_\alpha$. Take matrix units e_{ij} in A_α with the property that $qe_{ij} = \lambda_i e_{ij}$.

Proposition

We have, with 1_α and $1_{\overline{\alpha}}$ the identities in A_α and $A_{\overline{\alpha}}$ respectively and $c = \sum_i \lambda_i$,

$$\Delta(h)(1_{\overline{\alpha}} \otimes 1_\alpha) = \frac{1}{c} \sum_{i,j} \lambda_i \, S(e_{ij}) \otimes e_{ji}.$$

The idempotent $\Delta(h)(1_{\overline{\alpha}} \otimes 1_\alpha)$ is a separability idempotent in $A_{\overline{\alpha}} \otimes A_\alpha$.

The proof is using that $S^2(e_{ij}) = \lambda_i^{-1} \lambda_j e_{ij}$.

It is instructive to verify a few formulas.

We have e.g.

$$(S \otimes \iota)(\Delta(h)(1_{\overline{\alpha}} \otimes 1_\alpha)) = \frac{1}{c}\sum_{i,j} \lambda_i S^2(e_{ij}) \otimes e_{ji}$$

$$= \frac{1}{c}\sum_{i,j} \lambda_i \lambda_i^{-1} \lambda_j e_{ij} \otimes e_{ji} = \frac{1}{c}\sum_{i,j} \lambda_j e_{ij} \otimes e_{ji}$$

and if we apply the multiplication map on the right hand side we get

$$\frac{1}{c}\sum_{i,j} \lambda_j e_{ii} = \sum_i e_{ii} = 1_\alpha.$$

Also

$$(1 \otimes e_{rs})\Delta(h)(1_{\overline{\alpha}} \otimes 1_\alpha) = \frac{1}{c}\sum_{i,j} \lambda_i\, S(e_{ij}) \otimes e_{rs}e_{ji} = \frac{1}{c}\sum_i \lambda_i\, S(e_{is}) \otimes e_{ri}$$

$$(S(e_{rs}) \otimes \iota)\Delta(h)(1_{\overline{\alpha}} \otimes 1_\alpha) = \frac{1}{c}\sum_{i,j} \lambda_i (S(e_{rs})S(e_{ij}) \otimes e_{ji} = \frac{1}{c}\sum_i \lambda_i S(e_{is}) \otimes e_{ri}.$$

The left and right integral

From the formula for $\Delta(h)$ we can read the left and right integral φ and ψ on A.

Proposition

Fix an index α and let ω_α be the standard trace on A_α. Let q be an invertible element in $M(A)$ satisfying $aq = qS^2(a)$ for all a. Then, for $a \in A_\alpha$,

$$\varphi(a) = \omega_\alpha(q)\omega_\alpha(aq^{-1}) \quad \text{and} \quad \psi(a) = \omega_\alpha(q^{-1})\omega(aq).$$

We can verify

$$\begin{aligned}((\iota \otimes \varphi)\Delta(h))1_{\overline{\alpha}} &= \frac{1}{c}\sum_{i,j} \lambda_i S(e_{ij})\varphi(e_{ji}) \\ &= \frac{1}{c}\sum_{i} \lambda_i S(e_{ii}) c\lambda_i^{-1} \\ &= S(1_\alpha) = 1_{\overline{\alpha}}\end{aligned}$$

The dual of a discrete quantum group

The dual of A is the space B of linear functionals on A of the form $\varphi(\,\cdot\, a)$ with $a \in A$. As a space it is again the direct sum of the spaces A_α. For the product and the coproduct we have the following

$$\langle \Delta(a), b \otimes b' \rangle = \langle a, b'b \rangle \quad \text{and} \quad \langle a \otimes a', \Delta(b) \rangle = \langle aa', b \rangle.$$

For the involution we have

$$\langle a, b^* \rangle = \langle S(a^*), b \rangle^- \quad \text{and} \quad \langle a^*, b \rangle = \langle a, S(b)^* \rangle^-.$$

Proposition

The pair (B, Δ) is a Hopf *-algebra with an integral φ_B given by $\varphi_B(b) = \langle h, b \rangle$. It satisfies

$$\varphi_B(bb^*) = \varphi_A(aa^*) \quad \textit{when} \quad b = \varphi_A(a\,\cdot\,).$$

Hence it is a compact quantum group.

The dual of a compact quantum group

Now let B be a compact quantum group. Here it is a Hopf *-algebra with a positive integral φ_B. The dual A is the space of linear functionals on B of the form $\varphi_B(\,\cdot\, b)$. We have the same formulas for the product, the involution and the coproduct on A as before.

Proposition

There is a cointegral h in A given by $\langle h, b\rangle = \varphi_B(b)$. For the integrals we have now

$$\varphi_A(a^* a) = \varphi_B(b^* b) \quad \text{when} \quad a = \varphi_B(\,\cdot\, b).$$

The main issue to prove that we get a discrete quantum group is to show that A is a direct sum of matrix algebras. This follows from the following property.

Lemma

For all $a \in A$ the spaces Aa and aA are finite-dimensional.

This result is a consequence of the existence of a cointegral.

The quantum group $su_q(2)$ as a Hopf *-algebra

Start with the Hopf *-algebra, obtained as a deformation of the enveloping algebra of the Lie algebra of $SU(2)$, due to Jimbo.

Proposition

Take a real number $\lambda > 1$. Let $\mathfrak{A}$ be the unital *-algebra generated by elements q, e, f, were q is invertible and self-adjoint, $e^* = f$, and satisfying

$$qe = \lambda eq, \quad qf = \lambda^{-1} fq \quad \text{and} \quad ef - fe = \frac{1}{\lambda - \lambda^{-1}}(q^2 - q^{-2}).$$

Then it is a Hopf *-algebra for the coproduct defined by

$$\Delta(q) = q \otimes q \quad \text{and} \quad \Delta(e) = q \otimes e + e \otimes q^{-1}.$$

In order to make a discrete quantum group of this, we need enough finite-dimensional representations.

The quantum group $su_q(2)$ as a discrete quantum group

The finite-dimensional irreducible *-representations π_n are labeled by $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Each representation is characterized by its highest weight vector ξ satisfying

$$\pi_n(q)\xi = \lambda^n \xi \quad \text{and} \quad \pi_n(e)\xi = 0.$$

The dimension of π_n is $2n + 1$.

Proposition

Given two indices n, m, the *-representation $x \mapsto (\pi_n \otimes \pi_m)\Delta(x)$ is a direct sum of the representations π_k where

$$k = |n - m|, |n - m| + 1, \ldots, n + m.$$

The pair (A, Δ_A)

Let A be the direct sum of the matrix algebras $\pi_n(\mathfrak{A})$ where $n = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$.

Proposition

Define $\pi : \mathfrak{A} \to M(A)$ by $(\pi(x))_n = \pi_n(x)$. Then π is a non-degenerate *-homomorphism.

Proposition

There is coproduct $\Delta_A : A \to M(A \otimes A))$ defined and characterized by $\Delta_A(\pi(x)) = (\pi \otimes \pi)\Delta(x)$ for all $x \in \mathfrak{A}$.

The main point is the following. Given indices n, m denote by J the set of indices that appear in the tensor product representation $x \mapsto (\pi_n \otimes \pi_m)\Delta(x)$. Then $(\pi_n \otimes \pi_m)\Delta(x)$ only depends on $\pi_J(x)$, defined as $\sum_{k\in J} \pi_k(x)$. Moreover $x \in \mathfrak{A} \to \pi_J(x)$ has range equal to all of $\sum_{k\in J} \oplus A_k$.

Theorem

The pair (A, Δ_A) is a discrete quantum group.

The dual of the discrete quantum group (A, Δ_A)

Recall that the dual B of A is the space of linear functionals on A of the form $\varphi(\,\cdot\, a)$ with $a \in A$. Consider elements u_{ij} defined by $\langle a, u_{ij} \rangle = a_{ij}$ when $a \in A_{\frac{1}{2}}$. It defines a unitary 2×2 matrix in B. These elements generate B and $\Delta_B(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

Theorem

Let (C, Δ_C) be the quantum $SU_q(2)$-group as defined by Woronwicz with $q = \lambda^{-1}$. If we use α and γ for the standard generators of C, there is an surjective *-homomorphism θ of C to B given by

$$\theta(\alpha) = u_{11} \quad \text{and} \quad \theta(\gamma) = u_{12}.$$

It is compatible with the coproducts.

It is expected that θ is also injective so that we recover the quantum $SU_q(2)$ as the dual of the discrete quantum group we have constructed.

Conclusions

- We have presented a new treatment of discrete quantum groups.
- My main point is that discrete quantum groups should be studied in their own right, not as duals of compact quantum groups.
- We see that much of the information is contained already in the element $\Delta(h)$ in $M(A \otimes A)$.
- Its properties follow from the fact that it is a separability idempotent.
- We have illustrated the approach by constructing the quantum $su_q(2)$ from the Jimbo Hopf *-algebra. This is not entirely trivial.

Some references

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